Passing to the limit in (5.7) as $n \to \infty$, we obtain the estimates (4.4) with $\mathrm{const} = A_3$, which completes the proof of the method of two-scale expansions.

The extension of the results of this paper to the case when $I = (I^1, I^2, \ldots, I^s)$, $s > 1$, is trivial. For the multifrequency case $\varphi = (\varphi^1, \varphi^2, \ldots, \varphi^s)$, $s > 1$ there is no such simple and complete theory as in the case of $s = 1$.

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ON THE CONDITIONS FOR THE EXISTENCE OF THE REDUCING CHAPLYGIN FACTOR*

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The problem of the existence of a reducing Chaplygin factor (RCF) for non-holonomic systems with k degrees of freedom is discussed. By introducing additional coordinates, a class of non-holonomic systems for which the RCF method is applicable in a widened configuration space is distinguished. For comparison, the corresponding conditions in quasi-coordinates are given. The existence of an RCF for one of the equivalent non-holonomic systems is studied.

1. Formulation of the problem. S.A. Chaplygin formulated the conditions under which non-holonomic systems with two degrees of fredom can have a reducing factor (see /1/). Using the equations in admissible vectors, Chaplygin's ideas were extended to systems which have k degrees of freedom, /2/. The present paper continues the investigations initiated in /2/.

Let us recall from /2/ some of the equations necessary for our discussion. We assume for brevity that the indices $\lambda, \mu, \nu, \varkappa, \rho, \ldots$ take values from 1 to n; a, b, c, d from 1 to k; and $p, q, r, \ldots$ from k to n.

By means of

$$d\tau = N(q^\eta)\, dt \tag{1.1}$$

the equations of motion of a non-holonomic system in admissible vectors,

$$\frac{d}{dt}\left(\frac{\partial\theta}{\partial s^{\cdot a}}\right) - \frac{\partial\theta}{\partial q^\varkappa}\alpha_a^{\ \varkappa} - \Delta_{a,bc}s^{\cdot b}s^{\cdot c} = \frac{\partial U}{\partial q^\varkappa}\alpha_a^{\ \varkappa} \tag{1.2}$$

is changed to the form

$$\frac{d}{d\tau}\left(\frac{\partial(\theta)}{\partial s'^{a}}\right) - \frac{\partial(\theta)}{\partial q^\varkappa}\alpha_a^{\ \varkappa} = \frac{\partial U}{\partial q^\varkappa}\alpha_a^{\ \varkappa} \tag{1.3}$$

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$$(2\theta = G_{ab}s^{\cdot a}s^{\cdot b} = N^2 G_{a'b'}s'^{a}s'^{b} = 2\,(\theta),\ G_{ab} = g_{\lambda\mu}\alpha_a{}^{\lambda}\alpha_b{}^{\mu})$$

The coefficients $g_{\lambda\mu}$ are determined from the expressions for the kinetic energy $T = {}^1/_2 g_{\lambda\mu}q^{\cdot\lambda}q^{\cdot\mu}$, and the force function is denoted by $U\,(q^a)$.

2. The RCF method for a non-holonomic system with k degrees of freedom.

The admissible vectors have the form

$$\begin{aligned} &\alpha_1\,(1,\ 0,\ \ldots,\ 0,\ \omega_1{}^{k+1},\ \ldots,\ \omega_1{}^n)\\ &\alpha_2\,(0,\ 1,\ \ldots,\ 0,\ \omega_2{}^{k+1},\ \ldots,\ \omega_2{}^n)\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ &\alpha_k\,(0,\ 0,\ \ldots,\ 1,\ \omega_k{}^{k+1},\ \ldots,\ \omega_k{}^n)\end{aligned} \tag{2.1}$$

where $\omega_a{}^p\,(q^{\nu})$ are determined from the constraint equation

$$q^{\cdot p} = \omega_a{}^p q^{\cdot a} \tag{2.2}$$

With the above assumptions, Eqs.(1.3) can be presented in the form of the Lagrange equations

$$\frac{d}{d\tau}\frac{\partial\,(\theta)}{\partial q^{a'}} - \frac{\partial\,(\theta)}{\partial q^a} = \frac{\partial u}{\partial q^a} \tag{2.3}$$

The conditions for the existence of an RCF for a system with k degrees of freedom is

$$\begin{aligned} \Pi_{a,\,bc} &= \frac{\partial \ln N}{\partial q^{\varkappa}}\alpha_c{}^{\varkappa}G_{ab} - \frac{\partial \ln N}{\partial q^{\varkappa}}\alpha_b{}^{\varkappa}G_{ac}\\ &- 2\frac{\partial \ln N}{\partial q^{\varkappa}}\alpha_a{}^{\varkappa}G_{bc} - \Omega_{a,\,bc} = 0\\ \Omega_{a,\,bc} &= \Gamma_{c,\,ab} - \Gamma_{c,\,ba} + \Gamma_{b,\,ac} - \Gamma_{b,\,ca}\\ \Gamma_{a,\,cb} &= \Gamma_{\varkappa,\,\mu\nu}\alpha_a{}^{\varkappa}\alpha_b{}^{\mu}\alpha_c{}^{\nu} - g_{\lambda\mu}\alpha_a{}^{\lambda}\frac{\partial\alpha_b{}^{\mu}}{\partial q^{\nu}}\alpha_c{}^{\nu}\end{aligned} \tag{2.4}$$

As a consequence we obtain from them the equations

$$\begin{aligned} (1-k)\frac{\partial \ln N}{\partial q^{\varkappa}}\alpha_a{}^{\varkappa} &= \frac{1}{2}\Omega_a\\ \Omega_a &= \Omega_{a,\,bc}G^{bc}\end{aligned} \tag{2.5}$$

For $k = 2$, the number of Eqs.(2.4) and (2.5) is the same: there are two (this case was considered by Chaplygin).

Theorem 1. By changing the variables $q^{a'} = q^{a'}\,(q^a)$, $\det\|A_a{}^{b'}\| \neq 0$, the objects $\Pi_{a,\,bc}$ are transformed in accordance with the rule

$$\begin{aligned} &\Pi_{a',\,b'c'} = \Pi_{a,\,bc}A_{a'}{}^{a}A_{b'}{}^{b}C_{c'}{}^{c}\\ &(A_{a'}{}^{a} = \partial q^a/\partial q^{a'},\ A_b{}^{b'} = \partial q^{b'}/\partial q^b)\end{aligned}$$

By a direct check we can establish that

$$\begin{aligned} &\frac{\partial \ln N}{\partial q^{\varkappa}}\alpha_{a'}^{\varkappa} = \frac{\partial \ln N}{\partial q^{\varkappa}}\alpha_a{}^{\varkappa}A_{a'}^{a}\\ &G_{a'b'} = G_{ab}A_{a'}^{a}A_{b'}^{b},\quad \Omega_{a',\,b'c'} = \Omega_{a,\,bc}A_{a'}^{a}A_{b'}^{b}A_{c'}{}^{c}\end{aligned}$$

This proves the theorem.

Corollary. If a system with k degrees of freedom allows of the existence of an RCF, that is, if it satisfies the condition $\Pi_{a,\,bc} = 0$, then after a change of variables we obtain $\Pi_{a',\,b'c'} = 0$, and therefore the RCF of this system in the new variables is $N\,(q^{a'})$.

Let us put $\Lambda_{cb}{}^a = \Gamma_{bc}{}^a$. From the formulae transforming $\Lambda_{cb}{}^a$, using the above change of variables we obtain

$$\Lambda_{c'b'}^{a'} = \Lambda_{cb}^{a}A_a^{a'}A_{b'}^{b}A_{c'}{}^{c} + A_b^{a'}\frac{\partial A_{b'}^{b}}{\partial q^{c'}}$$

Hence it follows that $\Lambda_{cb}{}^a$ determine the affine connectivity in the space with torsion $S_{cb}{}^a = {}^1/_2\,(\Lambda_{cb}{}^a - \Lambda_{bc}{}^a)$. (see /3/). Using the results obtained in /3/, and the formulae

$$\nabla_a G_{bc} = \Omega_{a,\,bc} = \frac{\partial G_{bc}}{\partial q^a} - \Lambda_{ab}^{e}G_{ec} - \Lambda_{ac}^{e}G_{be},\quad \frac{\partial G_{bc}}{\partial q^a} = \Gamma_{b,\,ac} - \Gamma_{c,\,ab}$$

we can write (2.4) and (2.5) as follows:

$$\nabla_a G_{bc} = \nabla_c \ln NG_{ab} + \nabla_b \ln NG_{ac} - 2\nabla_a \ln NG_{bc} \tag{2.6}$$
$$(1-k)\nabla_a \ln N = {}^1/{}_2\Omega_a \tag{2.7}$$

On differentiating (2.6), and alternating the indices d and a, after a few operations we obtain

$$\nabla_d\nabla_a G_{bc} - \nabla_a\nabla_d G_{bc} = \Phi$$
$$\Phi_{dabc} = \nabla_d\nabla_c \ln NG_{ab} + \nabla_d\nabla_b \ln NG_{ac} - 2\nabla_d\nabla_a \ln NG_{bc} -$$
$$\nabla_a\nabla_c \ln NG_{db} - \nabla_a\nabla_b \ln NG_{dc} + 2\nabla_a\nabla_d \ln NG_{bc} +$$
$$\nabla_c \ln N\nabla_a \ln NG_{db} + \nabla_b \ln N\nabla_a \ln NG_{dc} - \nabla_c \ln N\nabla_d \ln NG_{ab} -$$
$$\nabla_b \ln N\nabla_d \ln NG_{ac}$$

Hence it follows that

$$-R^e_{dab}G_{ec} - R^e_{dac}G_{be} - 2S^e_{da}\nabla_e G_{bc} = \Phi_{dabc}$$

After performing the convolution of both sides of the above equalities with G^{bc}, we have

$$-2R^b_{dab} = 2(1-k)(\nabla_d\nabla_a \ln N - \nabla_a\nabla_d \ln N) + 2S^e_{da}\Omega_e \tag{2.8}$$

On differentiating (2.7) we find

$$2(1-k)(\nabla_d\nabla_a \ln N - \nabla_a\nabla_d \ln N) = \nabla_d\Omega_a - \nabla_a\Omega_d$$

The conditions for system (2.7) to be integrable can also be written in the form

$$-2S^e_{da}\Omega_e = \nabla_d\Omega_a - \nabla_a\Omega_d$$

Substituting them into (2.8), we have

$$R^b_{dab} = 0 \tag{2.9}$$

These conditions are satisfied when and only when system (2.7) is integrable. The Riemann-Christoffel tensor is found from the formula

$$R^d_{abc} = \frac{\partial \Lambda^d_{bc}}{\partial q^a} - \frac{\partial \Lambda^d_{ac}}{\partial q^b} + \Lambda^d_{ae}\Lambda^e_{bc} - \Lambda^d_{be}\Lambda^e_{ac} \tag{2.10}$$

Thus we have proved the following theorems.

Theorem 2. For $k = 2$ the necessary and sufficient invariant conditions for an RCF to exist are conditions (2.9).

When $k > 2$, only conditions (2.9) are necessary.

Theorem 3. For $k > 2$, the necessary and sufficient conditions for the existence of an RCF is the simultaneous satisfaction of conditions (2.9) and the following relations:

$$2(1-k)\Omega_{a,bc} = \Omega_c G_{ab} + \Omega_b G_{ac} - 2\Omega_a G_{bc} \tag{2.11}$$

where Ω_a is the gradient vector determined from (2.7), whose existence is ensured by the satisfaction of conditions (2.9).

Let us replace the system of admissible vectors $\alpha_a^\varkappa$ by the system $\beta_{a'}^\varkappa = \gamma_{a'}^a\alpha_a^\varkappa$ when $\det\|\gamma_{a'}^a\| \neq 0$. Using the results obtained in /2/, after some reduction we obtain

$$\Pi_{a',b'c'} = \Pi_{a,bc}\gamma^a_{a'}\gamma^b_{b'}\gamma_{c'}{}^c + \chi_{a',b'c'}$$
$$\chi_{a',b'c'} = G_{bc}\gamma_{c'}{}^c\left[\frac{\partial\gamma^b_{b'}}{\partial q^\varkappa}\alpha_a{}^\varkappa\gamma^a_{a'} - \frac{\partial\gamma^b_{a'}}{\partial q^\varkappa}\alpha_a{}^\varkappa\gamma^a_{b'}\right] + G_{bc}\gamma^c_{b'}\left[\frac{\partial\gamma^b_{c'}}{\partial q^\varkappa}\alpha_a{}^\varkappa\gamma^a_{a'} - \frac{\partial\gamma^b_{a'}}{\partial q^\varkappa}\alpha_a{}^\varkappa\gamma_{c'}{}^a\right] \tag{2.12}$$

Assuming that the system has an RCF, that is $\Pi_{a,bc} = 0$, we find

$$\Pi_{a',b'c'} = \chi_{a',b'c'} \tag{2.13}$$

Conditions (2.13) are referred to as the conditions for the existence of an RCF in the quasicoordinates (see /4, 5/). It was established in /5/ that for $k = 2$ the conditions derived in /4/ are incorrect. The correct conditions, obtained in /5/, are identical with (2.13).

It is clear from (2.12) that a case exists where $\Pi_{a',b'c'} = 0$ although $\Pi_{a,bc} \neq 0$. After the change $\beta_{a'}^\varkappa = \gamma_{a'}{}^a\alpha_a{}^\varkappa$, the admissible vectors $\beta_{a'}^\varkappa$, are not of the form (2.1).

Example. Consider a dynamic non-holonomic system with three degrees of freedom, whose double kinetic energy and the constraint equations have the form

$$2T = (q^{\cdot 1})^2 + (q^{\cdot 2})^2 + (q^{\cdot 3})^2 + (q^{\cdot 4})^2 + (q^{\cdot 5})^2$$
$$q^{\cdot 4} = q^{\cdot 2}\operatorname{tg} q^1, \quad q^{\cdot 5} = q^{\cdot 3}\operatorname{tg} q^1$$

and there are no outer active forces.

On substituting the expressions for $q^{\cdot 4}$ and $q^{\cdot 5}$ into $2T$, we obtain

$$2\theta = (q^{\cdot 1})^2 + [(q^{\cdot 2})^2 + (q^{\cdot 3})^2]/\cos^2 q^1$$

We can satisfy ourselves that

$$\Omega_{1,22} = \Omega_{1,33} = 2\sin q^1/\cos^3 q^1, \quad \Omega_{2,12} = \Omega_{3,13} = -\sin q^1/\cos^3 q^1$$

The remaining quantities $\Omega_{a,bc}$ are zero. From (2.5) we find $\Omega_1 = 4\,\mathrm{tg}\, q^1$, $\Omega_2 = \Omega_3 = 0$, $N = \cos q^1$. The conditions (2.11) are satisfied, therefore the function $N = \cos q^1$ is an RCF of the system.

3. The RCF method in a widened configuration space. Let the vectors $\beta_{a'}{}^{\varkappa}$, after the change, have the following form:

$$\beta_{1'}(1, 0, \ldots, 0, \omega_{1'}^{l+1}, \omega_{1'}^{l+2}, \ldots, \omega_{1'}{}^{n}), \ldots, \beta_{l'}(0, 0, \ldots, 1, \omega_{l'}^{l+1}, \omega_{l'}^{l+2}, \ldots, \omega_{l'}{}^{n}), \beta_{(l+1)'}(0, 0, \ldots, 0, \omega_{(l+1)'}^{l+1}, \omega_{(l+1)'}^{l+2}, \ldots, \omega_{(l+1)'}^{n}), \ldots, \beta_{k'}(0, 0, \ldots, 0, \omega_{k'}^{l+1}, \omega_{k'}^{l+2}, \ldots, \omega_{k'}^{n}) \tag{3.1}$$

From the relations above we see that $q^1, q^2, \ldots, q^l$ $(0 \leqslant l < k)$ are coordinates. We widen the configuration space (see /6/) by introducing the auxiliary coordinates $\pi^{l+1}, \ldots, \pi^k$, assume that $q^{1'} = q^1, \ldots, q^{l'} = q^l$, $q^{(l+1)'} = \pi^{l+1}, \ldots, q^{k'} = \pi^k$, $q^{(k+1)'} = q^{l+1}, \ldots, q^{(n-l+k)'} = q^n$, and and introduce the notation $2T' = g_{i'j'}q^{\cdot i'}q^{\cdot j'}$ $(i', j' = 1, 2, \ldots, n-l+k)$. We have supposed up to now that $g_{\lambda\mu}$, $\omega_a{}^p$ and U are functions of the coordinates $q^1, q^2, \ldots, q^k$. Here and below we shall require that these functions depend on $q^1, q^2, \ldots, q^l$ only. In the notation of the admissible vectors

$$\alpha_{1'}(1, 0, \ldots, 0, \omega_{1'}^{l+1}, \ldots, \omega_{1'}{}^{n}), \alpha_{2'}(0, 1, \ldots, 0, \omega_{2'}^{l+1}, \ldots, \omega_{2'}{}^{n}), \ldots, \alpha_{k'}(0, 0, \ldots, 1, \omega_{k'}^{l+1}, \ldots, \omega_{k'}{}^{n}) \tag{3.2}$$

the coordinates which occupy places from $l+1$ to k correspond to the variables $\pi^{l-1}, \ldots, \pi^k$. In the widened $(n-l+k)$-dimensional space the admissible vectors (3.2) are already of the form (2.1).

The equations of motion of a non-holonomic system in this space are (see /7/):

$$\left(\frac{d}{dt}\frac{\partial L'}{\partial q^{\cdot j'}} - \frac{\partial L'}{\partial q^{j'}}\right)\alpha_{a'}^{j'} = 0, \quad L' = T' + U \tag{3.3}$$

$$q^{\cdot j'} = \alpha_{a'}^{j'} q^{\cdot a'} \tag{3.4}$$

If $\omega_{\varkappa}{}^p q^{\cdot\varkappa} = 0$ is an equation of a certain non-holonomic connectedness of the initial system (1.2), then $\omega_{\varkappa}{}^p\alpha_a{}^{\varkappa} = 0$ and $\omega_{\varkappa}{}^p\beta_a{}^{\varkappa} = 0$. By (3.3), all terms for which $j' = l+1, \ldots, k$, are identically zero. Hence

$$\left(\frac{d}{dt}\frac{\partial L}{\partial q^{\cdot\varkappa}} - \frac{\partial L}{\partial q^{\varkappa}}\right)\beta_{a'}^{\varkappa} = 0, \quad L = T + U \tag{3.5}$$

If we take from (3.4) all equations except those with numbers $j' = l+1, \ldots, k$, we obtain

$$q^{\cdot\varkappa} = \beta_{a'}^{\varkappa} q^{\cdot a'} = \gamma_{a'}^{a}\alpha_a{}^{\varkappa} q^{\cdot a'} \tag{3.6}$$

Taking the convolution of both sides with respect to $\omega_{\varkappa}{}^p$, we arrive at the expression

$$\omega_{\varkappa}{}^p q^{\cdot\varkappa} = 0 \tag{3.7}$$

Obviously, Eqs.(3.3), (3.4) are equivalent to (3.5), (3.7), and, in addition, to the $k-l$ equations which are linear with respect to the derivatives of the coordinates. The last equations set $k-l$ additional non-holonomic constraints

$$\Omega_{i'}{}^{\varepsilon} q^{\cdot i'} = 0, \quad \varepsilon = n+1, \ldots, n+k-l \tag{3.8}$$

Eqs.(3.5), (3.7) define the motions of the output system (1.2).

The conditions for the existence of an RCF for the Eqs.(3.3), (3.4),

$$\frac{\partial \ln N}{\partial q^{j'}}\alpha_{c'}{}^{j'}G_{a'b'} + \frac{\partial \ln N}{\partial q^{j'}}\alpha_{b'}{}^{j'}G_{a'c'} - 2\frac{\partial \ln N}{\partial q^{j'}}\alpha_{a'}^{j'}G_{b'c'} = \Omega_{a',b'c'} \tag{3.9}$$

can be obtained as in /2/, by operating in the widened configuration space. The difference is that the rank $\|g_{i'j'}\| = n$, since the row and column elements of this matrix, with numbers from $l+1$ to k, equals zero.

The problem of finding the reducing factor $N(q^1, \ldots, q^l, \pi^{l+1}, \ldots, \pi^k)$ is equivalent to that discussed in /2/, the difference being that the matrix $\| g_{i'j'} \|$ degenerates. The equations for determining the RCF,

$$(1-k)\frac{\partial \ln N}{\partial q^m} = \frac{1}{2}\Omega_m, \quad m = 1, 2, \ldots, l \tag{3.10}$$

$$(1-k)\frac{\partial \ln N}{\partial \pi^s} = \frac{1}{2}\Omega_s, \quad s = l+1, \ldots, k$$

are obtained from the findings of /2/. It is desirable if possible to integrate Eqs.(3.10), and to satisfy conditions (3.9) since this will ensure that the RCF $N(q^1, \ldots, q^l, \pi^{l+1}, \ldots, \pi^k)$ is found.

Considering the above assumption, we write Eqs.(1.3) in a widened configuration space as follows:

$$\frac{d}{d\tau}\left(\frac{\partial(\theta')}{\partial q^{m'}}\right) - \frac{\partial(\theta')}{\partial q^m} = \frac{\partial U}{\partial q^m}, \quad m = 1, 2, \ldots, l \tag{3.11}$$

$$\frac{d}{d\tau}\left(\frac{\partial(\theta')}{\partial \pi^{s'}}\right) - \frac{\partial(\theta')}{\partial \pi^s} = 0, \quad s = l+1, \ldots, k \tag{3.12}$$

$$2\theta' = G_{a'b'}s^{\cdot a'}s^{\cdot b'} = N^2 G_{a'b'}s'^{a'}s'^{b'} = 2(\theta')$$

$$s^{1'} = q^1, \ldots, s^{l'} = q^l; \quad s^{(l+1)'} = \pi^{l+1}, \ldots, s^{k'} = \pi^k$$

In obtaining (3.11) and (3.12) we use the fact that $g_{\lambda\mu}$, $\omega_a{}^p$ and U are functions of $q^1, q^2, \ldots, q^l$ only. Requirements of this kind are met in /4/. As was noted in /5/, they lead to false conclusions since quasicoordinates were used in /4/. If the discussion is conducted in a widened configuration space, a class of non-holonomic systems for which the RCF method is applicable can be selected.

The non-holonomic system discussed can be replaced by an equivalent non-degenerate system (see /8,9/). It will have the Lagrangian

$$L^* = L - {}^1\!/_2 \delta_{\varepsilon\chi} \Omega_{i'}{}^{\varepsilon} \Omega_{j'}{}^{\chi} q^{\cdot i'} q^{\cdot j'} \tag{3.13}$$

where $\delta_{\varepsilon\chi}$ are Kronecker deltas. The system is subject to non-holonomic constraints (3.7) and (3.8). According to /8,9/, in this case neither the equations of motion in the admissible vectors nor conditions (3.9) will vary.

Theorems 2 and 3 will then be formulated as follows.

Theorem 4. For $k = 2$, the necessary and sufficient invariant conditions for the existence of an RCF in a widened configuration space are

$$R^{b'}_{d'a'b'} = 0 \tag{3.14}$$

Theorem 5. For $k > 2$, the necessary and sufficient conditions for the existence of an RCF in a widened configuration space are expressions (3.14) and (3.9).

In the process of forming the Riemann-Christoffel tensor we must use the values of $\Gamma^{a'}_{c'b'}$ calculated in the widened configuration space. The connection between $R^{d'}_{a'b'c'}$ and R^{d}_{abc} can be found from the formula

$$\Gamma^{a'}_{c'b'} = \Gamma^{a}_{cb}\gamma_{c'}{}^{c}\gamma_{b'}{}^{b}\gamma_{a}^{a'} - \gamma_{b}^{a'}\frac{\partial \gamma_{b'}^{b}}{\partial q^{\rho}}\alpha_c{}^{\rho}\gamma_{c'}{}^{c} \tag{3.15}$$

(see /7/). After certain operations we have

$$R^{d'}_{a'b'c'} = R^{d}_{abc}\gamma_{a'}^{a}\gamma_{b'}^{b}\gamma_{c'}{}^{c}\gamma_{d}^{d'} + R^{*d'}_{a'b'c'} \tag{3.16}$$

Formulae (3.16) and (2.13) make it possible to formulate Theorems 2 and 3 in quasi-coordinates.

Theorem 6. For $k = 2$ the necessary and sufficient conditions for the existence of an RCF in quasicoordinates are the conditions

$$R^{c'}_{a'b'c'} = R^{*c'}_{a'b'c'} \tag{3.17}$$

Theorem 7. For $k > 2$, the necessary and sufficient conditions for the existence of an RCF in quasicoordinates are the conditions (3.17) and (2.13).

Generally, $R^{*c'}_{a'b'c'} \neq 0$. This confirms once more that there is a case where the method is in applicable in an initial space, but is applicable in a configuration space.

4. The equivalent non-holonomic systems and the problem of the existence of an RCF. The reduction of the equations of a non-holonomic system to the Lagrange type of equations, based on Helmholtz's conditions was considered in /10,11/. The reduction was achieved either indirectly or after a suitable change of the right-hand side of the equations of motion. Two non-holonomic systems are referred to as equivalent when they have the same trajectories on the manifold determined from the constraint equations (see /8,9/). This is equivalent to the requirement that these systems are subject to the same constraints and have the same equations of motion which are widened with respect to the highest derivatives

$$s^{\cdot\cdot a} + \Gamma^a_{bc} s^{\cdot b} s^{\cdot c} = F^a, \quad s^{\cdot\cdot a} + \Gamma^{*a}_{bc} s^{\cdot b} s^{\cdot c} = F^{*a}$$

where $q^{\cdot\varkappa} = \alpha^\varkappa \dot{s}^a$ (see /7/). The conditions of equivalence are expressed as follows (/8/):

$$\Gamma^a_{bc} + \Gamma^a_{cb} = \Gamma^{*a}_{bc} + \Gamma^{*a}_{cb}, \quad F^a = F^{*a} \tag{4.1}$$

Let us consider a non-holonomic system with a Lagrange function L. We denote by $L_l = L + L_1$ the Lagrange function equivalent to the non-holonomic system, where

$$2L_1 = \theta_{bc} s^{\cdot b} s^{\cdot c} + \theta_{bp} s^{\cdot b} s^{\cdot p} + \theta_{pb} s^{\cdot b} s^{\cdot p} + 2V \tag{4.2}$$
$$s^{\cdot a} = \omega_\varkappa^a q^{\cdot\varkappa}, \quad s^{\cdot p} = \omega_\varkappa^p q^{\cdot\varkappa}, \quad \omega_\varkappa^a = G^{ab} \alpha_b^\varkappa g_{\varkappa\lambda}$$

In the Chaplygin systems, θ_{ab}, θ_{pb} and V are functions of q^a only. The modified conditions of equivalence (4.1) (see /9/) are expressed as

$$2\nabla_c \theta_{ab} = 4S^e_{bc}\theta_{ea} + 4S^e_{ac}\theta_{eb} + M^r_{ca}\theta_{br} + M^r_{cb}\theta_{ar} \tag{4.3}$$
$$G^{ec}\theta_{ae} \frac{\partial U}{\partial q^\varkappa} \alpha_c^\varkappa = \frac{\partial V}{\partial q^\varkappa} \alpha_a^\varkappa$$
$$\left(M^r_{ca} = \left(\frac{\partial \omega_\varkappa^r}{\partial q^\rho} - \frac{\partial \omega_\rho^r}{\partial q^\varkappa} \right) \alpha_c^\varkappa \alpha_a^\rho \right)$$

In an equivalent system functions N and G_{ab} correspond to N^* and $G_{ab}^* = G_{ab} + \theta_{ab}$, and and the conditions for the existence of an RCF have the form

$$\nabla_c \ln N^* G^*_{ab} + \nabla_b \ln N^* G^*_{ac} - 2\nabla_a \ln N^* G^*_{bc} = \tag{4.4}$$
$$2\nabla_a G^*_{bc} + 2S^e_{ab} G^*_{ec} + 2S^e_{ac} G^*_{eb}$$

On substituting (2.10) into (2.9), we obtain

$$R^c_{abc} = \partial\Gamma^c_{cb} / \partial q^a - \partial\Gamma^c_{ca} / \partial q^b = 0 \tag{4.5}$$

On the other hand,

$$2\frac{\partial\Gamma^c_{bc}}{\partial q^a} = \frac{\partial}{\partial q^a}\left[\frac{\partial G_{de}}{\partial q^b} G^{de}\right] = \frac{\partial}{\partial q^a}\left[\frac{1}{D}\frac{\partial D}{\partial G_{de}}\frac{\partial G_{de}}{\partial q_b}\right] = \frac{\partial^2 \ln D}{\partial q^a\,\partial q^b}$$

$(D = \det \| G_{de} \|)$. Therefore, $\partial\Gamma_{bc}^c / \partial q^a - \partial\Gamma_{ac}^c / \partial q^b = 0$. Adding this quantity to the right-hand side of (4.4) we find

$$R^c_{abc} = \partial(\Gamma^c_{cb} - \Gamma^c_{bc}) / \partial q^a - \partial(\Gamma^c_{ca} - \Gamma^c_{ac}) / \partial q^b = 0$$

Now, taking into account the first relation of (4.1), we formulate the following theorem.

Theorem 8. The necessary conditions for the existence of an RCF for the whole class of equivalent non-holonomic systems are conditions (2.9) or, correspondingly, (3.14).

Theorems 3, 5 and 7 give the sufficient conditions for the existence of an RCF for the class of equivalent non-holonomic systems. To find the necessary and sufficient conditions we must look into the equestion of the compatibility of (4.3) and (4.4). Let us consider the solution of this problem in the case when $N^* = 1$. The Eq.(4.4) has the form

$$\nabla_a\theta_{bc} = -\nabla_a G_{bc} - S^e_{ba}(G_{ec} + \theta_{ec}) - S^e_{ca}(G_{eb} + \theta_{eb}) \tag{4.6}$$

Using (4.6) we eliminate $\nabla_c\theta_{ab}$ from the first equation of (4.3). This yields

$$2\nabla_c G_{ab} + 2S^e_{bc}(\theta_{ea} - G_{ea}) + 2S^e_{ac}(\theta_{eb} - G_{eb}) + M^r_{ca}\theta_{br} + M^r_{cb}\theta_{ar} = 0 \tag{4.7}$$

System (4.6) can be solved separately. The quantities which satisfy system (4.6) are determined from the integrability condition. Eq.(4.7) and the second equation of (4.3), like the integrability condition, are linear in θ_{cb} and θ_{bp}, and their common solution is equivalent to the problem stated. It was shown in /11/ that the equations of motion of a sphere on a horizontal plane without slip, after a suitable change of thier right-hand sides take the form of the Lagrange equations for a holonomic system. Hence it follows that the reducing factor $N^* = 1$ exists (the existence of this example was noted by Chaplygin, in /1/).

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FLOW OF A MULTILAYER IDEAL INCOMPRESSIBLE AND HEAVY FLUID PAST A BODY*

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The two-dimensional steady flow of a layered fluid past a body with discontinuous stratification is disucssed. The number of layers in finite, and the channel which has a horizontal floor is open. To study the flow behind the body, a hypothesis on the possibility of approximating the velocity profile at the body boundary by that which arises in weightless flow (see /1,2/) is postulated. A boundary value problem for a second-order elliptic equation in combined Euler-Lagrange variables is formulated. The problem is formulated in a rectilinear band with a separation, and under the conditions of consistency, on a finite number of parallel straight lines which correspond to the separation boundary. The introduction of a measure which gives rise to a monotonic density distribution in a non-perturbed flow, makes it possible to reduce the boundary value problem to the symmetrization of Fredholm-type kernels. The linearized equation is solved by Fourier methods.

The results obtained in /3/ are amplified: it is shown that for any specified Froude number, the corresponding homogeneous integral equation has only a finite number of positive eigenvalues to which the oscillation modes correspond. It is also shown that if the flow velocity is close to one of a denumerable set of propagation velocities of long-wave modes, the corresponding harmonic becomes stronger because of the resonance.

1. Formulation of the problem. Consider the two-dimensional steady flow of an ideal incompressible heavy stratified fluid past a body T_0: $(|x| \leqslant l,\ y_-(x) \leqslant y \leqslant y_+(x))$, where $y_+(x)$ and $y_-(x)$ are known functions which define the body shape. The Ox axis is directed along the horizontal floor of the channel, and the Oy axis runs vertically upwards (see the figure). At the boundaries $y_k(x)$ of the layer T_k, the density ρ and the tangential component of the velocity $\mathbf{V}$ suffer a discontinuity, and the pressure p and the normal

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